Some characteristics of the atmosphere during an adiabatic process*

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Abstract Some important characteristics of the atmosphere during an adiabatic process are investigated, which include the invariability of atmospheric entropy range and local surface potential temperature, the conservation of the atmospheric mass intervened between any isentropic surface and the ground, and the isentropic surface intersecting with the ground. The analysis shows that the atmospheric reference state (ARS) for investigation on available potential energy (APE) should be defined objectively as the state which could be approached from the existing atmosphere by adiabatic adjustment, and be related to initial atmospheric state before adjustment. For the initial atmosphere state at any time, its corresponding ARS is different from the one at another time. Based on the above-mentioned conclusions, the reference state proposed by Lorenz cannot be obtained physically, so a new conception, the conditional minimum total potential energy, is put forward in order to objectively investigate atmospheric APE.

Keywords: available potential energy, adiabatic, isentropic surface, atmospheric reference state, conditional minimum total potential energy.

The studies on available potential energy (APE) always play an important role in atmospheric energetics. The terminology of APE was first introduced for the general circulation by Margules[1], and later redefined by Lorenz[2,3]. Furthermore, approximate and exact formulas for calculating APE and its generation were promoted, but those derivations are traditionally based on the assumptions that hydrostatic balance prevails, the atmosphere is stably stratified everywhere, the latent energy does not contribute to the internal energy, and surface topography can be ignored. There have been many approximations developed for handling those assumptions. For example, Dutton and Johnson derived a "more exact" equation by eliminating the assumption of hydrostatic balance[4], and Lorenz developed moist available energy by considering the vapor process[5], and Taylor took surface topography into account[6]. Lorenz's APE concept has also been extended to study the variation of APE in a limited region during the process of formation of storms[7—10]. In recent years, the theory of APE has new development in many aspects[11—13], and has also been applied widely to the studies of atmospheric and oceanic energetics[14—18].

In the work on theoretical derivation and calculation of APE, to define a suitable atmospheric reference state (ARS) by redistributing atmospheric mass under thermodynamically reversible adiabatic process is of importance. In Lorenz's first ARS without topography, the atmosphere is barotropic, horizontal, stably stratified, and in minimum TPE[2,3,5]. Further researches on ARS have been done, such as the case with topography[6], etc. Actually, such distributions of ARS have been directly designed physically, whereas some characteristics of atmosphere on isentropic surfaces (ISs) under adiabatic condition have not yet been mathematically investigated. Moreover, how ARS can be obtained from an existing atmospheric state in terms of adiabatic adjustment, and whether ARS designed physically can be approached, have not been proved mathematically. Evidently, solving these essential problems are very important for the comprehension of ARS and APE, which will be examined in the present paper.

1 Variation of local surface potential temperature during an adiabatic process

APE is defined by Lorenz as the difference between the total potential energy (TPE) of an existing

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atmospheric state and some suitably defined reference state[2,3]. Usually the reference state represents the minimum total potential energy (MTPE), which can be attained by redistributing atmospheric mass under thermodynamically reversible adiabatic condition. According to Lorenz's definition of ARS, the atmosphere is barotropic, horizontal, stably stratified, and in the state of MTPE. If there is no surface topography, the local surface potential temperature of the ARS is homogeneous and just identical to the minimum of potential temperature before adiabatic adjustment. Taylor put forward two particular ARSs with surface topography considered[6]. One has the minimum entropy due to taking surface topography into ARS directly, and the other has uniform surface pressure, undulate isentropic surfaces with topography, and the minimum enthalpy. We can see that for different purposes, the definition of ARS can be chosen arbitrarily. Therefore, it is necessary to examine whether the assumed ARS can be approached really in an existing atmospheric state by adiabatic adjustment. So far rigorous derivations have not been seen yet.

We first give boundary conditions in spherical isentropic coordinate $(\lambda, \varphi, \theta, t)$ (where λ, φ indicate geographical longitude and latitude, respectively), $\lambda \in [0, 2\pi]$, $\varphi \in [-\pi/2, \pi/2]$, θ is potential temperature and t the time) as follows[19,20]:

Upper boundary: when $\theta = \theta_T = \text{const}$,

$$\dot{\theta}_T = 0; \tag{1}$$

lower boundary: when $\theta = \theta_S(\lambda, \varphi, t)$,

$$\dot{\theta}_S = \frac{\partial \theta_S}{\partial t} + \frac{u_S}{a\cos\varphi}\frac{\partial \theta_S}{\partial \lambda} + \frac{v_S}{a}\frac{\partial \theta_S}{\partial \varphi}; \tag{2}$$

where θ_T, θ_S are potential temperature at the top of atmosphere and on the surface of the earth, respectively. u_S, v_S are zonal and meridional components of wind at the surface, respectively, and a is mean radius of the earth.

As we have known, under adiabatic condition, the potential temperature of air parcel remains constant, viz. $d\theta = 0$. Since vertical θ velocity $\dot{\theta} = d\theta/dt = 0$, ISs are substantial surfaces. During the process of adiabatic adjustment, substantial surfaces can neither vanish nor be created, and there is no motion which can pass through ISs. So we obtain

Characteristic 1. During an adiabatic process, the range of atmospheric entropy or potential temperature is invariant.

This characteristic can be easily documented. Given that the range of atmospheric potential temperature is originally expressed as $[\theta_{min}(0), \theta_{max}(0)]$ and written as $[\theta_{min}(t), \theta_{max}(t)]$ at any time t. If $\theta_{min}(t) \neq \theta_{min}(0)$, $\theta_{max}(t) \neq \theta_{max}(0)$, there will exist motion passed through isentropic surface during the process of adiabatic adjustment, which is evidently contradictory to the above-mentioned.

Under adiabatic condition, lower boundary condition (2) can be written as $\dot{\theta}_S = 0$. If there is no boundary flow[19] (namely $\mathbf{V}_{hS} = 0$, corresponding to viscous lower boundary condition), or in more common sense, there is no entropy advection on the surface ($\mathbf{V}_{hS} \cdot \nabla \theta_S = 0$), we can obtain that local change of θ_S is zero. So it can be obtained that

Characteristic 2. During an adiabatic process, if there is no flow across the boundary, or there is no entropy advection on the surface, the surface potential temperature will locally remain invariable.

This implies that local potential temperature on the surface always keeps its original value in the existing atmosphere. Under the condition of no boundary flow and no surface entropy advection, local surface potential temperature does not vary with the process of atmospheric adiabatic adjustment.

Based on Characteristics 1 and 2, it is easy to obtain two simple corollaries as follows:

Corollary 1. During an adiabatic process, if there is no boundary flow, or there is no entropy advection on the surface, the range of atmospheric entropy or potential temperature in any vertical air column will be constant.

Corollary 2. During an adiabatic process, if there is no boundary flow, or there is no entropy advection on the surface, the position of intersection lines between isentropic surfaces and the ground will never change.

It is clear that the fore-mentioned derivations can be reasonable with or without topography. This shows that during the process of adiabatic adjustment, the potential temperature at any place on the surface remains invariable from the existing atmosphere to its ARS. Therefore, after adjustment, the ISs intersected with ground are not always superposable with iso-geopotential surfaces, which indicates that the previous definitions of ARS can be unobjective.

In addition, because atmospheric state is diabatically changing at every time, surface potential temperature varies with time, which makes the local surface potential temperature after adiabatic adjustment different at every time. This implies that ARS should be different with respect to the existing atmosphere state at different time. In the following discussion, we can see that the invariability of local surface potential temperature is quite important for the atmospheric adiabatic process due to the mass conservation restriction.

2 Atmospheric mass conservation on isentropic surfaces during an adiabatic process

Based on the invariability of local surface potential temperature under adiabatic condition, now some characteristics of the atmosphere on the ISs during the process of adiabatic adjustment will be further discussed. The atmospheric mass conservation is a basic law as we know. For the characteristics of mass conservation, the previous work focused on the case that ISs do not intersect with ground[2—4]. But the case that ISs intersect with ground, which is quite essential to understanding the features of ISs, has not been documented clearly.

In the spherical isentropic coordinate, the mass continuity equation may be expressed as:

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \theta}\right)\right]_\theta+\frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial \lambda}\left(u\frac{\partial p}{\partial \theta}\right)\right]_\theta+\frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial \varphi}\left(v\cos\varphi\frac{\partial p}{\partial \theta}\right)\right]_\theta+\frac{\partial}{\partial \theta}\left(\dot{\theta}\frac{\partial p}{\partial \theta}\right)=0, \tag{3}$$

where p is pressure, u and v are zonal and meridional components of wind, respectively. Here, the pressure tendency on the top of atmosphere is ignored for convenience, namely $\partial p_{\theta_T}/\partial t=0$[19,20].

Considering the general case that any isentropic surface θ_1 intersects with ground, let σ_1 denote the overground section of the isentropic surface θ_1 (namely, if $(\lambda,\varphi)\in\sigma_1, \theta_1\geqslant\theta_S(\lambda,\varphi)$), and let σ_{1S} denote the ground section of the isentropic surface θ_1 (namely, if $(\lambda,\varphi)\in\sigma_{1S}, \theta_1\leqslant\theta_S(\lambda,\varphi)$), and let Γ denote a collection of intersection lines between isentropic surface θ_1 and ground (namely the boundary $\Gamma=\sigma_1\cap\sigma_{1S}$). We can integrate Eq.(3) from the closed curved surface $\sigma_G=\sigma_1\cup\sigma_{1S}$ to the atmospheric top with the upper boundary condition, hence it follows that

$$\begin{aligned}&\iint_{\sigma_1}\int_{\theta_1}^{\theta_T}\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial t}\right)_\theta d\theta d\sigma+\iint_{\sigma_{1S}}\int_{\theta_S}^{\theta_T}\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial t}\right)_\theta d\theta d\sigma\\&+\iint_{\sigma_1}\int_{\theta_1}^{\theta_T}\frac{1}{a\cos\varphi}\left\{\left[\frac{\partial}{\partial\lambda}\left(u\frac{\partial p}{\partial\theta}\right)\right]_\theta+\left[\frac{\partial}{\partial\varphi}\left(v\cos\varphi\frac{\partial p}{\partial\theta}\right)\right]_\theta\right\}d\theta d\sigma\\&+\iint_{\sigma_{1S}}\int_{\theta_S}^{\theta_T}\frac{1}{a\cos\varphi}\left\{\left[\frac{\partial}{\partial\lambda}\left(u\frac{\partial p}{\partial\theta}\right)\right]_\theta+\left[\frac{\partial}{\partial\varphi}\left(v\cos\varphi\frac{\partial p}{\partial\theta}\right)\right]_\theta\right\}d\theta d\sigma\\&-\iint_{\sigma_1}\dot{\theta}_1\left(\frac{\partial p}{\partial\theta}\right)_{\theta_1}d\sigma-\iint_{\sigma_{1S}}\dot{\theta}_S\left(\frac{\partial p}{\partial\theta}\right)_{\theta_S}d\sigma=0,\end{aligned} \tag{4}$$

in which $d\sigma=a^2\cos\varphi d\lambda d\varphi$ is the integral element in horizontal direction. Since

$$\left(\frac{\partial p}{\partial t}\right)_{\theta_S}=\frac{\partial p_S}{\partial t}-\left(\frac{\partial p}{\partial\theta}\right)_{\theta_S}\frac{\partial\theta_S}{\partial t},$$

with the aid of atmospheric upper boundary condition, the first two terms of Eq.(4) can be rewritten as

$$\begin{aligned}&\iint_{\sigma_1}\int_{\theta_1}^{\theta_T}\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial t}\right)_\theta d\theta d\sigma+\iint_{\sigma_{1S}}\int_{\theta_S}^{\theta_T}\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial t}\right)_\theta d\theta d\sigma\\&=-\iint_{\sigma_1}\left(\frac{\partial p}{\partial t}\right)_{\theta_1}d\sigma-\iint_{\sigma_{1S}}\left(\frac{\partial p}{\partial t}\right)_{\theta_S}d\sigma\\&=-\iint_{\sigma_G}\frac{\partial p}{\partial t}d\sigma+\iint_{\sigma_{1S}}\left(\frac{\partial p}{\partial\theta}\right)_{\theta_S}\frac{\partial\theta_S}{\partial t}d\sigma.\end{aligned} \tag{5}$$

Thus, the integral of pressure tendency on the curved surface σ_G is obtained. For any variable A, there exists $\iint_{\sigma_G}A d\sigma=\iint_{\sigma_1}A_{\theta_1}d\sigma+\iint_{\sigma_{1S}}A_S d\sigma$, in which A_{θ_1} is the value of variable A on the isentropic surface θ_1, and A_S the value of A on the ground.

For the fourth term of Eq.(4), with the law of taking derivative of variable-limit integral we can obtain

$$\frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\lambda}\left(\int_{\theta_S}^{\theta_T}u\frac{\partial p}{\partial\theta}d\theta\right)\right]_\theta=\int_{\theta_S}^{\theta_T}\frac{1}{a\cos\varphi}\cdot\left[\frac{\partial}{\partial\lambda}\left(u\frac{\partial p}{\partial\theta}\right)\right]_\theta d\theta-\frac{u_S}{a\cos\varphi}\frac{\partial\theta_S}{\partial\lambda}\left(\frac{\partial p}{\partial\theta}\right)_{\theta_S}, \tag{6}$$

$$\frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\varphi}\int_{\theta_S}^{\theta_T}v\cos\varphi\frac{\partial p}{\partial\theta}d\theta\right]_\theta=\int_{\theta_S}^{\theta_T}\frac{1}{a\cos\varphi}$$

$$\cdot \left[\frac{\partial}{\partial \varphi}\left(v \frac{\partial p}{\partial \theta}\right)\right]_{\theta} \mathrm{d}\theta - \frac{v_S}{a}\frac{\partial \theta_S}{\partial \varphi}\left(\frac{\partial p}{\partial \theta}\right)_{\theta_S}. \quad (7)$$

Substituting (5)—(7) into Eq.(4), we obtain

$$-\iint\limits_{\sigma_G} \frac{\partial p}{\partial t}\mathrm{d}\sigma + \iint\limits_{\sigma_{1S}} \left(\frac{\partial p}{\partial \theta}\right)_{\theta_S} \frac{\partial \theta_S}{\partial t}\mathrm{d}\sigma + \iint\limits_{\sigma_G} \frac{1}{a\cos\varphi}\left\{\left[\frac{\partial}{\partial \lambda}\left(\int_{\theta_W}^{\theta_T} u\frac{\partial p}{\partial \theta}\mathrm{d}\theta\right)\right]_{\theta} + \left[\frac{\partial}{\partial \varphi}\left(\int_{\theta_W}^{\theta_T} v\cos\varphi \frac{\partial p}{\partial \theta}\mathrm{d}\theta\right)\right]_{\theta}\right\}\mathrm{d}\sigma + \iint\limits_{\sigma_{1S}} \left(\frac{u_S}{a\cos\varphi}\frac{\partial \theta_S}{\partial \lambda} + \frac{v_S}{a}\frac{\partial \theta_S}{\partial \varphi}\right)\left(\frac{\partial p}{\partial \theta}\right)_{\theta_S}\mathrm{d}\sigma - \iint\limits_{\sigma_1} \dot{\theta}_1\left(\frac{\partial p}{\partial \theta}\right)_{\theta_1}\mathrm{d}\sigma - \iint\limits_{\sigma_{1S}} \dot{\theta}_S\left(\frac{\partial p}{\partial \theta}\right)_{\theta_S}\mathrm{d}\sigma = 0,$$

in which, when $(\lambda, \varphi) \in \sigma_1$, $\theta_W = \theta_1$; when $(\lambda, \varphi) \in \sigma_{1S}$, $\theta_W = \theta_S$. With the lower boundary condition (2), we can further obtain

$$\iint\limits_{\sigma_G} \frac{\partial p}{\partial t}\mathrm{d}\sigma = -\iint\limits_{\sigma_1} \dot{\theta}_1\left(\frac{\partial p}{\partial \theta}\right)_{\theta_1}\mathrm{d}\sigma. \quad (8)$$

For an atmospheric adiabatic process, the right-hand side of Eq.(8) is equal to zero, thus

$$\iint\limits_{\sigma_G} \frac{\partial p}{\partial t}\mathrm{d}\sigma = 0. \quad (9)$$

In Eq.(9), whether the order between integral and differential operators can be exchanged will entirely depend on the fact whether the boundary $\Gamma = \sigma_1 \cap \sigma_{1S}$ varies with time. As Corollary 2 has shown, when there is no surface entropy advection, the positions of the intersection lines (namely the boundary Γ) between ISs and ground will remain invariable. At this moment, we can obtain the characteristics of atmospheric mass conservation over a closed curved surface σ_G by Eq. (9):

$$\frac{\partial}{\partial t}\left(\iint\limits_{\sigma_G} \frac{p}{g}\mathrm{d}\sigma\right) = 0, \quad (10)$$

where g is the acceleration of gravity. In particular, if σ_{1S} is null set Φ, then the isentropic surface has no intersection with ground, and $\sigma_1 = \sigma_G$. Eq. (10) is still proper. On the other hand, if σ_1 is null set Φ, $\sigma_S = \sigma_G$, and Eq.(10) represents the conservation of the whole atmospheric mass. Thus it follows that

Characteristic 3. During an adiabatic process, atmospheric mass over the isentropic surfaces which do not intersect with ground will maintain conservation; when isentropic surfaces intersect with ground, under the condition of no surface entropy advection, the atmospheric mass over a closed curved surface σ_G can also maintain conservation.

Therefore, it can be seen that:

Corollary 3. During an adiabatic process, if there is no surface entropy advection, (i) the atmospheric mass between any two continuous isentropic surfaces will maintain conservation; (ii) the atmospheric mass between any isentropic surface and ground will maintain conservation.

As the above derivations have shown, for the case that there exists surface entropy advection, which means surface potential temperature varies with time, Eq. (10) cannot be obtained from (9). Hence, the mass between any IS and ground can not maintain conservation if IS intersects with ground. Controlled by the law of mass conservation, either no surface entropy advection or the invariability of local surface potential temperature is an essential physical restriction for the adiabatic process in the atmosphere.

3 Conditional minimum total potential energy

According to Characteristic 3, the atmospheric mass enclosed with ISs and ground maintains conservation, but its distribution may change. So there exists a minimum total potential energy (MTPE). Furthermore, for the initial atmospheric state at any time, its corresponding ARS is different from the one at another time. Thus, such a MTPE should be conditional and depends on the initial distribution of atmospheric state at different time, which is called the conditional minimum total potential energy (CMTPE) here. During an adiabatic process, not only the potential temperature of air parcel is invariable, but also the mass maintains conservation. Restricted by the law of mass conservation, the positions of intersection lines between ISs and ground are also fixed and unchangeable with time. Hence, in Lorenz's definition, the ideal ARS with horizontal ISs may be never approached by adiabatic adjustment, and its corresponding TPE is just a lower limit of TPE among different ARSs, which may be called the absolute MTPE.

In the expression of Lorenz's APE[2,3], the integrand is the difference between TPE and MTPE in unit air column. In order to guarantee consistent isentropic surface, which means vertical integral limits at any column keep the same during the process of adiabatic adjustment, Lorenz has made an extension (named Lorenz extension here): when $\theta < \theta_S(\lambda, \varphi)$, $p(\lambda, \varphi, \theta) = p_S(\lambda, \varphi)$. Thus, the pressure of every spot under ground is substituted by corresponding surface pressure, which makes all of ISs closed. As we know, the ISs intersected with ground cannot reach horizontal states by adiabatic adjustment under the restriction of mass conservation. So Lorenz extension is almost impossible to be applied physically, although it is mathematically feasible. Actually this extension is unnecessary, because local surface potential temperature has invariability under adiabatic condition. So any ARS, which cannot be approached by adiabatic adjustment, is not objective.

On the other hand, any atmospheric state with different distribution of surface potential temperature should have its corresponding ARS approached by adiabatic adjustment. Because the positions of intersection lines between ISs and ground are fixed, theoretically the ARS after adjustment exists uniquely. If the MTPE is regarded as a criterion, the APE can be determined in terms of the difference between the TPE of existing atmosphere and the variational minimum of TPE under the restriction of boundary condition with fixed local potential temperature. This will be meaningful and is worth further studies.

References

1 Margules M. Über die energie der stürme. Jahrb Zentralanst Meteor, 1903, 40: 1—26.

2 Lorenz E. N. Available potential energy and the maintenance of the general circulation. Tellus, 1955, 7(2): 157—167.

3 Lorenz E. N. The Nature and Theory of the General Circulation of the Atmosphere. Geneva: World Meteorological Organization Publication, 1967, 97—107.

4 Dutton J. A. and Johnson D. R. The theory of available potential energy and a variational approach to atmospheric energetics. Advances in Geophys., 1967, 12: 333—436.

5 Lorenz E. N. Available energy and the maintenance of a moist circulation. Tellus, 1978, 30: 15—31.

6 Taylor K. E. Formulas for calculating available potential energy over uneven topography. Tellus, 1979, 31: 236—245.

7 Smith P. J. A computational study of the energetics of a limited region of the atmosphere. Tellus, 1969, 21(2): 193—201.

8 Smith P. J. On the contribution of a limited region to the global energy budget. Tellus, 1969, 21(2): 202—207.

9 Johnson D. R. The available potential energy of storms. J. Atmos. Sci., 1970, 27: 727—741.

10 Edmon H. J. Jr. A reexamination of limited-area available potential energy budget equations. J. Atmos. Sci., 1978, 35: 1655—1659.

11 Boer G. J. Zonal and eddy forms of the available potential energy equations in pressure coordinates. Tellus, 1975, 27(5): 433—442.

12 Siegmund P. The generation of available potential energy, according to Lorenz' exact and approximate equations. Tellus, 1994, 46A(5): 566—582.

13 Shepherd G. A unified theory of available potential energy. Atmos. Ocean., 1993, 31: 1—26.

14 Oort A. H., Ascher S. C., Levitus S. et al. New estimates of the available potential energy in the world ocean. J. Geophys. Res., 1989, 94: 3187—3200.

15 Huang R. Mixing and available potential energy in a boussinesq ocean. J. Phys. Oceanogr., 1998, 28: 669—678.

16 Blanchard D. O. Assessing the vertical distribution of convective available potential energy. Weather and Forecasting, 1998, 13: 870—887.

17 Gettelman A., Seidel D. J., Wheeler M. C. et al. Multidecadal trends in tropical convective available potential energy. J. Geophys. Res., 2002, 107(D21): 4606.

18 DeMott C. A. and Randall D. A. Observed variations of tropical convective available potential energy. J. Geophys. Res., 2004, 109: D02102.

19 Liu S. K. and Liu S. D. The Atmospheric Dynamics. Beijing: Peking University Press, 1991, 126—129, 151—153.

20 Kasahara A. Various vertical coordinate systems used for numerical weather prediction. Mon. Wea. Rev., 1974, 102: 509—522.